

OPTIMAL SZEGÖ-WEINBERGER TYPE INEQUALITIES

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ABSTRACT. Denote with $\mu_1(\Omega; e^{h(|x|)})$ the first nontrivial eigenvalue of the Neumann problem

$$\begin{cases} -\operatorname{div}(e^{h(|x|)} \nabla u) = \mu e^{h(|x|)} u & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded and Lipschitz domain in \mathbb{R}^N . Under suitable assumption on h we prove that the ball centered at the origin is the unique set maximizing $\mu_1(\Omega; e^{h(|x|)})$ among all Lipschitz bounded domains Ω of \mathbb{R}^N of prescribed $e^{h(|x|)} dx$ -measure and symmetric about the origin. Moreover, an example in the model case $h(|x|) = |x|^2$, shows that, in general, the assumption on the symmetry of the domain cannot be dropped. In the one-dimensional case, i.e. when Ω reduces to an interval (a, b) , we consider a wide class of weights (including both Gaussian and anti-Gaussian). We then describe the behavior of the eigenvalue as the interval (a, b) slides along the x -axis keeping fixed its weighted length.

Key words: Weighted Neumann eigenvalues; Symmetrization; Isoperimetric estimates
MSC: 35B45; 35P15; 35J70

1. INTRODUCTION

In [22] Kornhauser and Stakgold made a famous conjecture: among all planar simply connected domains, with fixed Lebesgue measure the first nontrivial eigenvalue of the Neumann Laplacian achieves its maximum value if and only if Ω is a disk. This conjecture was proved by Szegő in [27]. In [29] Weinberger generalized this result to any bounded smooth domain of \mathbb{R}^N . Adapting Weinberger arguments, similar inequalities for spaces of constant sectional curvature have been derived (see, e.g., [2] and [11]). Further, it is proved in [23] that the first nonzero Neumann eigenvalue is maximal for the equilateral triangle among all triangles of given perimeter, and hence among all triangles of given area.

For broad surveys of isoperimetric eigenvalue inequalities, one can consult, for instance, the monographs of Bandle [3], Henrot [18] and [19], Kesavan [21], and the survey paper by Ashbaugh [1].

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In this paper we derive some sharp Szegő-Weinberger type inequalities for the first non-trivial eigenvalue, $\mu_1(\Omega; e^{h(|x|)})$, of the following class of problems

$$\begin{cases} -\operatorname{div}(e^{h(|x|)} \nabla u) = \mu e^{h(|x|)} u & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

Here and in the sequel, Ω will denote a bounded domain in \mathbb{R}^N with Lipschitz boundary and ν the outward normal to $\partial\Omega$. Since the degeneracy of the operator is given in terms of the radial function $e^{h(|x|)}$, it appears natural to let Ω vary in the class of sets having prescribed γ_h -measure, where

$$d\gamma_h = e^{h(|x|)} dx, \text{ with } x \in \mathbb{R}^N. \quad (1.2)$$

Recently, in [13], it has been proved that among all Lipschitz domains Ω in \mathbb{R}^N , which are symmetric about the origin and have fixed Gaussian measure, $\mu_1(\Omega; e^{-|x|^2/2})$ achieves its maximum value if and only if Ω is the Euclidean ball. In the same paper it has been shown that $\mu_1((a, b); e^{-t^2/2})$ is minimal when the interval (a, b) reduces to a half-line and maximal when it is centered at the origin, and it is strictly monotone as (a, b) slides between these extreme positions.

The first part of the present paper deals with a class of weighted eigenvalue problems in the form

$$\begin{cases} -(u'q)' = \mu uq & \text{in } (a, b) \\ u'(a) = u'(b) = 0, \end{cases} \quad (1.3)$$

where $a, b \in \mathbb{R}$ with $a < b$ and

$$q(x) \in C^2(\mathbb{R}), \quad q(x) > 0 \quad \forall x \in \mathbb{R}. \quad (1.4)$$

We analyze the behavior of $\mu_1((a, b); q)$ as the interval (a, b) slides along the x -axis keeping fixed its q -length $(= \int_a^b q(x) dx)$. We prove that if $q(x) = q(|x|)$ and it is decreasing on \mathbb{R}_+ then $\mu_1((a, b); q)$ behaves like $\mu_1((a, b); e^{-t^2/2})$, while if $q(x)$ is increasing on \mathbb{R}_+ then $\mu_1((a, b); q)$ behaves in the opposite way. Note that the method used here is different and more general than the one of [13]. For the precise statement see Theorem 1.1 below. It treats the Gaussian and anti-Gaussian-type weights in a unified way. We emphasize that, in contrast to the case $N \geq 2$, no concavity assumptions are imposed on the weight function. This explains the different notation in (1.1) and (1.3).

Let $c := \int_{\mathbb{R}} q(t) dt \in (0, +\infty]$, and fix $d \in (0, c)$. We define $\bar{a} > 0$ such that

$$\int_{-\bar{a}}^{\bar{a}} q(t) dt = d$$

If $c < +\infty$, then we define a_+ by

$$\int_{a_+}^{+\infty} q(t) dt = d,$$

otherwise, put $a_+ = +\infty$.

We consider the function $b = b(a)$ defined on $(-\infty, a_+)$ such that

$$\int_a^{b(a)} q(t) dt = d.$$

Denote by $\mu_1(a)$ the eigenvalue $\mu_1((a, b(a)); q)$. Then we have the following

Theorem 1.1. *Assume that q satisfies (1.4) and is even. Then*

- (i) $\mu_1(a) = \mu_1(-b(a))$ for $a < a_+$,
- (ii) if $q'(x) \underset{(\leq)}{\geq} 0$ for $x \geq 0$, then $\frac{d}{da}\mu_1(a) \underset{(\leq)}{\geq} 0$ for $a > -\bar{a}$,
- (iii) if q is not constant on $(a, b(a))$ and $q'(x) \underset{(\leq)}{\geq} 0$ for $x \geq 0$, then $\frac{d}{da}\mu_1(a) \underset{(<)}{>} 0$ for $a > -\bar{a}$.

Now let us turn our attention to the N -dimensional problem (1.1).

We assume that the function $h(r)$ fulfills the following set of hypotheses

$$h(r) \in C^2([0, +\infty[), \quad h'(r) > -\frac{N-1}{r} \quad \forall r > 0, \quad h''(r) \geq 0 \quad \forall r \geq 0. \quad (1.5)$$

Our second main result is the following.

Theorem 1.2. *Assume that assumptions (1.5) are in force. Then the ball centered at the origin is the unique set maximizing $\mu_1(\Omega; e^{h(|x|)})$ among all Lipschitz bounded domains Ω of \mathbb{R}^N of prescribed γ_h -measure and symmetric about the origin. Moreover, if the assumption on the symmetry of the domain is dropped, then, in general, the thesis does not hold true.*

Finally note that Theorem 1.2 in particular applies to $\mu_1(\Omega; e^{|x|^2})$. Indeed, as model case we may choose $h(|x|) = |x|^2$. Hence our result gives information about the Neumann eigenvalues of the problem

$$-\Delta u - 2x \cdot \nabla u = \mu u,$$

which is widely studied in literature (see, e.g., [10] and [9]). For related results see also [4–6] and [16, 17].

The paper is organized as follows. Section 2 contains some results from the theory of weighted rearrangements along with the definition of the suitable Sobolev spaces naturally associated to problem (1.1). Section 3 is devoted to Theorem 1.1. By a suitable change of

variable, we firstly show that $\mu_1((a, b); q)$ coincides with $\lambda_1((a, b); q^{-1})$, the first Dirichlet eigenvalue of problem (1.3) with respect the weight q^{-1} . In turn, we observe that the following problems

$$P_q : \begin{cases} -\frac{d}{dx} \left(\frac{1}{q(x)} \frac{dv}{dx} \right) = \lambda \frac{1}{q(x)} v(x) & \text{in } (a, b) \\ v(a) = v(b) = 0 \end{cases} \quad \text{and} \quad P_m : \begin{cases} -\frac{d^2 w}{dy^2} = \lambda m(y) w(y) & \text{in } (\alpha, \beta) \\ w(\alpha) = w(\beta) = 0, \end{cases}$$

are isospectral, where $y =: F(x) := \int_0^x q(t) dt$, $\alpha := F(a)$, $\beta := F(b)$ and m is defined in (3.15). The advantages of studying P_m in place of P_q are twofold. First, since the function $F(x)$ pushes the measure $q(x)dx$ forward dy , as long as the interval (a, b) moves along the x -axis with fixed q -length, then (α, β) slides along the y -axis keeping fixed just its Lebesgue measure. Second, the new equation contains a weight only in the zero order term, so it nicely behaves under reflection with respect to $(\alpha + \beta)/2$. These two circumstances allow to evaluate the sign of the shape derivative of the first eigenvalue of P_m and hence of $\mu_1((a, b); q)$. In Section 4 we prove Theorem 1.2. To this aim, we first study problem (1.1) in the radial case, i.e., when $\Omega = B_R$, the ball centered at the origin of radius R . We deduce that $\mu_1(B_R; e^{h(|x|)})$ is an eigenvalue of multiplicity N , and a corresponding set of linearly independent eigenfunctions is $\left\{ w(|x|) \frac{x_i}{|x|}, i = 1, \dots, N \right\}$, with an appropriate function w . We then define $G(r) = w(r)$ for $0 \leq r \leq R$ and $G(r) = w(R)$ for $r > R$. Since the functions $G(|x|) \frac{x_i}{|x|}$, $(i = 1, \dots, N)$, have mean value zero, we may use them in the variational characterization of $\mu_1(\Omega; e^{h(|x|)})$. Then the result is achieved by symmetrization arguments.

2. NOTATION AND PRELIMINARY RESULTS

Now we recall a few definitions and properties about weighted rearrangement. For exhaustive treatment on this subject we refer, e.g., to [14], [20] and [26].

Throughout this paper, B_R will denote the ball of \mathbb{R}^N centered at the origin of radius R .

Let $u : x \in \Omega \rightarrow \mathbb{R}$ be a measurable function. We denote by $m(t)$ the distribution function of $|u(x)|$ with respect to γ_h -measure, defined in (1.2), i.e.

$$m(t) = \gamma_h(\{x \in \Omega : |u(x)| > t\}), \quad t \geq 0,$$

while the decreasing rearrangement and the increasing rearrangement of u are defined respectively by

$$u^*(s) = \inf \{t \geq 0 : m(t) \leq s\}, \quad s \in]0, \gamma_h(\Omega)]$$

and

$$u_*(s) = u^*(\gamma_h(\Omega) - s), \quad s \in [0, \gamma_h(\Omega)[.$$

Finally u^\star , the γ_h -rearrangement of u , is given by

$$u^\star(x) = u^\star(|x|) = u^*(\gamma_h(B_{|x|})), \quad x \in \Omega^\star,$$

where Ω^\star is the ball B_{r^\star} such that $\gamma_h(\Omega^\star) = \gamma_h(B_{r^\star})$. By its very definition u^\star is a radial and radially decreasing function. Since u and u^\star are γ_h -equimeasurable, Cavalieri's principle ensures

$$\|u\|_{L^p(\Omega; \gamma_h)} = \|u^\star\|_{L^p(\Omega^\star; \gamma_h)}, \quad \forall p \geq 1.$$

We will also make use of the Hardy-Littlewood inequality, which states that

$$\int_0^{\gamma_h(\Omega)} u^*(s) v_*(s) ds \leq \int_\Omega |u(x) v(x)| d\gamma_h \leq \int_0^{\gamma_h(\Omega)} u^*(s) v^*(s) ds. \quad (2.1)$$

Since Ω is a bounded domain, assumptions (1.5) ensures that

$$0 < c_1 < e^{h(|x|)} < c_2 < +\infty, \quad \text{in } \Omega, \quad (2.2)$$

for some constants c_1 and c_2 .

The natural functional space associated to problem (1.1) is the weighted Sobolev space defined as follows

$$H^1(\Omega; \gamma_h) = \{u \in W_{\text{loc}}^{1,1}(\Omega) : (u, |\nabla u|) \in L^2(\Omega; \gamma_h) \times L^2(\Omega; \gamma_h)\},$$

endowed with the norm

$$\|u\|_{H^1(\Omega; \gamma_h)}^2 = \|u\|_{L^2(\Omega; \gamma_h)}^2 + \| |\nabla u| \|_{L^2(\Omega; \gamma_h)}^2 = \int_\Omega u^2 d\gamma_h + \int_\Omega |\nabla u|^2 d\gamma_h. \quad (2.3)$$

By (2.2) one immediately infers that

$$u \in H^1(\Omega; \gamma_h) \iff u \in H^1(\Omega),$$

since the norm defined in (2.3) is equivalent to the usual norm in $H^1(\Omega)$. Hence we have that $H^1(\Omega; \gamma_h)$ is compactly embedded in $L^2(\Omega; \gamma_h)$. By standard theory on self-adjoint compact operator, $\mu_1(\Omega; e^{h(|x|)})$ admits the following well known variational characterization

$$\mu_1(\Omega; e^{h(|x|)}) = \min \left\{ \frac{\int_\Omega |\nabla v|^2 d\gamma_h}{\int_\Omega v^2 d\gamma_h} : v \in H^1(\Omega) \setminus \{0\}, \int_\Omega v d\gamma_h = 0 \right\}. \quad (2.4)$$

3. THE ONE-DIMENSIONAL CASE

Throughout this Section we will assume that condition (1.4) is fulfilled and that

$$-\infty < a < b < +\infty.$$

We consider the weighted Neumann eigenvalue problem

$$\begin{cases} -(u'q)' = \mu uq & \text{in } (a, b) \\ u'(a) = u'(b) = 0. \end{cases} \quad (3.1)$$

The first nontrivial eigenvalue of (3.1), $\mu_1((a, b); q)$, clearly fulfills

$$\mu_1((a, b); q) = \min \left\{ \frac{\int_a^b (u')^2 q \, dx}{\int_a^b u^2 q \, dx} : u \in H^1(a, b) \setminus \{0\}, \int_a^b u q \, dx = 0 \right\}. \quad (3.2)$$

Here we are interested in studying the behavior of $\mu_1((a, b); q)$ when the interval (a, b) slides along the x -axis, keeping fixed its weighted length $\int_a^b q(x) \, dx$.

Further, we consider the weighted Dirichlet eigenvalue problem

$$\begin{cases} -\left(v' \frac{1}{q}\right)' = \lambda v \frac{1}{q} & \text{in } (a, b) \\ v(a) = v(b) = 0, \end{cases} \quad (3.3)$$

and we denote its first eigenvalue by $\lambda_1((a, b); q^{-1})$. In the next Lemma, we will simply write μ_1 and λ_1 for $\mu_1((a, b); q)$ and $\lambda_1((a, b); q^{-1})$, respectively.

Lemma 3.1. *There holds*

$$\mu_1 = \lambda_1. \quad (3.4)$$

Further, if u_1 is an eigenfunction to problem (3.1) corresponding to μ_1 , then the function

$$v := u_1' q \quad (3.5)$$

is an eigenfunction to problem (3.3) with eigenvalue μ_1 . Finally, if v_1 is an eigenfunction to problem (3.3) corresponding to λ_1 , and if $x_0 \in (a, b)$ is such that $v_1'(x_0) = 0$, then

$$u(x) := \int_{x_0}^x v_1(t) \frac{1}{q(t)} dt \quad (3.6)$$

is an eigenfunction to problem (3.1) corresponding to λ_1 . Finally, there holds

$$u(x) = -\frac{v_1'(x)}{\lambda_1 q(x)}. \quad (3.7)$$

Proof. The equation in (3.1) can be clearly written in the following form

$$-u_1'' - \frac{q'}{q} u_1' = \mu_1 u_1$$

By differentiating we get

$$-u_1''' - \frac{q''}{q} u_1' + \frac{q'^2}{q^2} u_1' - \frac{q'}{q} u_1'' = \mu_1 u_1'. \quad (3.8)$$

Plugging (3.5) into (3.8) one obtains that

$$-\left(\frac{v'}{q}\right)' = -\frac{v''}{q} + \frac{v'}{q^2} q' = \mu_1 \frac{v}{q},$$

with $v(a) = v(b) = 0$. This means that v is an eigenfunction to problem (3.3) corresponding to μ_1 , and thus

$$\lambda_1 \leq \mu_1. \quad (3.9)$$

Next, let u be defined by (3.6). Then obviously

$$u'(a) = u'(b) = 0. \quad (3.10)$$

By definition v_1 fulfills the equation in problem (3.3) with $\lambda = \lambda_1$, i.e.

$$-\left(v_1' \frac{1}{q}\right)' = \lambda_1 v_1 \frac{1}{q} \quad (3.11)$$

Integrating equation (3.11) from x_0 to x and taking into account that $v_1'(x_0) = 0$ and (3.6), we obtain

$$-v_1' \frac{1}{q} = \lambda_1 u,$$

that is (3.7).

Now, since $v_1 = qu_1'$, we have $v_1' = q'u_1' + qu_1''$. This, together with (3.7) and (3.10), implies

$$\begin{cases} -(u'q)' = \lambda_1 uq & \text{in } (a, b) \\ u'(a) = u'(b) = 0. \end{cases}$$

Hence u is an eigenfunction to problem (3.1) with eigenvalue λ_1 , and

$$\lambda_1 \geq \mu_1. \quad (3.12)$$

Now (3.9) and (3.12) imply (3.4). The Lemma is proved. \square

Now we introduce a new independent variable for problem (3.3),

$$y =: F(x) := \int_0^x q(t) dt \quad \text{with } \alpha := F(a) \quad \text{and } \beta := F(b). \quad (3.13)$$

Let us consider the following eigenvalue problem

$$\begin{cases} -w'' = kmw & \text{in } (\alpha, \beta) \\ w(\alpha) = w(\beta) = 0, \end{cases} \quad (3.14)$$

where

$$m(y) := \frac{1}{[q(F^{-1}(y))]^2}. \quad (3.15)$$

We will denote by $k_n((\alpha, \beta); m)$ the sequence of eigenvalues to problem (3.14), arranged in increasing order. It is straightforward to verify that v_n is an eigenfunction to problem (3.3) corresponding to its n -th eigenvalue $\lambda_n((a, b); q^{-1})$ if and only if $w_n(y) := v_n(F^{-1}(y))$, is an

eigenfunction to the new problem (3.14) corresponding to its n -th eigenvalue $k_n((\alpha, \beta); m)$. Indeed, since $F^{-1}(y)$ is a strictly increasing function admitting a zero ($F^{-1}(y) = 0$ if and only if $y = 0$), we have that $w_n(y)$ and $v_n(x)$ have the same number of nodal domains, therefore (see, e.g., [15] Vol. 1 p. 454) there holds

$$k_n((\alpha, \beta); m) = \lambda_n((a, b); q^{-1}), \quad \forall n \in \mathbb{N}. \quad (3.16)$$

Lemma 3.2. *Let m be defined by (3.15) and even, that is $m(y) = m(-y) \forall y \in (-c/2, c/2)$, where $c = \int_{\mathbb{R}} q(t) dt$. Assume $\alpha + \beta > 0$. Let w_1 be the eigenfunction to problem (3.14) corresponding to $k_1((\alpha, \beta); m)$, normalized so that $w_1(y) > 0$ and*

$$\int_{\alpha}^{\beta} w_1^2(y) m(y) dy = 1.$$

(i) *If $m'(y) \geq 0$ for $y > 0$, then*

$$-w_1'(\beta) \geq w_1'(\alpha). \quad (3.17)$$

(ii) *If $m'(y) \leq 0$ for $y > 0$, then*

$$-w_1'(\beta) \leq w_1'(\alpha). \quad (3.18)$$

Moreover inequalities (3.17) and (3.18) are strict unless m is constant on (α, β) .

Proof. Setting $\bar{w}(y) := w_1(\alpha + \beta - y)$, it follows that

$$\begin{cases} -\bar{w}'' = k_1((\alpha, \beta); m) \bar{w}(y) m(\alpha + \beta - y) & \text{on } (\alpha, \beta) \\ \bar{w}(\alpha) = \bar{w}(\beta) = 0, \end{cases}$$

and

$$w_1\left(\frac{\alpha + \beta}{2}\right) = \bar{w}\left(\frac{\alpha + \beta}{2}\right).$$

Further, setting $W := \bar{w} - w_1$, we find

$$\begin{cases} -W'' = k_1((\alpha, \beta); m) m(y) W + g & \text{on } (\alpha, \frac{\alpha + \beta}{2}) \\ W(\alpha) = W(\frac{\alpha + \beta}{2}) = 0 \end{cases} \quad (3.19)$$

where

$$g(y) := k_1((\alpha, \beta); m) \bar{w}(y) (m(\alpha + \beta - y) - m(y)).$$

By the well known monotonicity property of the Dirichlet eigenvalues with respect to domains inclusion, we have that

$$k_1\left(\left(\alpha, \frac{\alpha + \beta}{2}\right); m\right) > k_1((\alpha, \beta); m).$$

From the maximum principle we obtain that the solution W of the boundary value problem (3.19), with g given, is unique. Moreover, $W \equiv 0$ for $g \equiv 0$, $W > 0$ for $g \geq 0, g \not\equiv 0$ and $W < 0$ for $g \leq 0, g \not\equiv 0$. Hence, using the strong maximum principle, we get

$$\begin{aligned} 0 &= W'(\alpha) = -w'_1(\alpha) - w'_1(\beta) \quad \text{for } g \equiv 0, \\ 0 &< W'(\alpha) = -w'_1(\alpha) - w'_1(\beta) \quad \text{for } g \geq 0, g \not\equiv 0 \quad \text{on } \left(\alpha, \frac{\alpha+\beta}{2}\right), \\ 0 &> W'(\alpha) = -w'_1(\alpha) - w'_1(\beta) \quad \text{for } g \leq 0, g \not\equiv 0 \quad \text{on } \left(\alpha, \frac{\alpha+\beta}{2}\right). \end{aligned}$$

Note that $g \geq 0$ ($g \leq 0$) on $(\alpha, \frac{\alpha+\beta}{2})$ if $m'(y) \geq 0$ ($m'(y) \leq 0$) for $y > 0$, and $g \equiv 0$ is possible in either case if $m = \text{const.}$ on (α, β) . This completes the proof of Lemma. \square

Let $\alpha, \beta \in \mathbb{R}$ be defined in (3.13), let $t > 0$, be such that $-c/2 < \alpha < \beta < \beta + t < c/2$, where $c = \int_{\mathbb{R}} q(t) dt$. We consider the following family of eigenvalue problems

$$\begin{cases} -\tilde{w}_{yy}(y, t) = k(t)m(y)\tilde{w}(y, t) & \text{on } (\alpha + t, \beta + t) \\ \tilde{w}(\alpha + t, t) = \tilde{w}(\beta + t, t) = 0. \end{cases} \quad (3.20)$$

Let

$$k_1(t) := k_1((\alpha + t, \beta + t); m) \quad (3.21)$$

be the first eigenvalue to problem (3.20), and denote with $\tilde{w}(y, t)$ the corresponding eigenfunction, normalized so that $\tilde{w}(y, t) > 0$ and

$$\int_{\alpha+t}^{\beta+t} \tilde{w}(y, t)^2 m(y) dy = 1. \quad (3.22)$$

Clearly

$$\tilde{w}(y, 0) = w_1(y),$$

where $w_1(y)$ is the function defined in Lemma 3.2.

Lemma 3.3. *Let $k_1(t)$ be defined by (3.21). It holds that*

$$k'_1(0) = -(\tilde{w}_y(\beta, 0))^2 + (\tilde{w}_y(\alpha, 0))^2 = -(w'_1(\beta))^2 + (w'_1(\alpha))^2. \quad (3.23)$$

Proof. By (3.22) we have

$$k_1(t) = \int_{\alpha+t}^{\beta+t} \tilde{w}_y(y, t)^2 dy,$$

by differentiating we find

$$k'_1(0) = 2 \int_{\alpha}^{\beta} \tilde{w}_y(y, 0) \tilde{w}_{ty}(y, 0) dy + (\tilde{w}_y(\beta, 0))^2 - (\tilde{w}_y(\alpha, 0))^2. \quad (3.24)$$

Moreover, from (3.22) it follows that

$$\int_{\alpha}^{\beta} \tilde{w}(y, t) \tilde{w}_t(y, 0) m(y) dy = 0. \quad (3.25)$$

Finally, a differentiation with respect to t of the equation in problem (3.20) gives

$$-\tilde{w}_{tyy}(y, 0) = k'_1(0)m(y)\tilde{w}(y, 0) + k_1(0)m(y)\tilde{w}_t(y, 0).$$

Multiplying the above equation with $\tilde{w}(y, 0)$ and integrating yields

$$\int_{\alpha}^{\beta} \tilde{w}_y(y, 0)\tilde{w}_{ty}(y, 0)dy = k'_1(0) \int_{\alpha}^{\beta} m(y)\tilde{w}(y, 0)^2 dy + k_1(0) \int_{\alpha}^{\beta} m(y)\tilde{w}_t(y, 0)\tilde{w}(y, 0)dy.$$

Using (3.22), (3.25) and integrating by parts, we have

$$\int_{\alpha}^{\beta} \tilde{w}_y(y, 0)\tilde{w}_{ty}(y, 0)dy = k'_1(0),$$

this, together with (3.24), gives the claim (3.23). \square

Lemma 3.4. *Let $\alpha + \beta > 0$ and $m(y)$ be an even function, and let $k_1(t)$ be given by (3.21).*

(i) *If $m'(y) \geq 0$ for $y > 0$, then*

$$k'_1(0) \leq 0. \quad (3.26)$$

(ii) *If $m'(y) \leq 0$ for $y > 0$, then*

$$k'_1(0) \geq 0. \quad (3.27)$$

Moreover, the inequalities (3.26), (3.27) are strict unless m is constant on (α, β) .

Proof. Lemma 3.3 tells us that

$$k'_1(0) = (w'_1(\alpha) + w'_1(\beta))(w'_1(\alpha) - w'_1(\beta)). \quad (3.28)$$

Since we are assuming that $w_1(y) > 0$ in (α, β) , it holds that

$$w'_1(\alpha) > 0 > w'_1(\beta). \quad (3.29)$$

The thesis follows from (3.28), (3.29) and Lemma 3.2. \square

Now let \bar{a} , a_+ , c and d the constants defined in the Introduction. Recalling that the function $b = b(a)$ is defined on $(-\infty, a_+)$ by the identity

$$\int_a^{b(a)} q(t) dt = d. \quad (3.30)$$

In the y variable (see (3.13)), the above condition obviously becomes

$$\beta = \beta(\alpha) = \alpha + d.$$

Now we are in position to prove the main result of this Section.

Proof of Theorem 1.1. Since $q(x)$ is an even function, we have that $b(-b(a)) = a$ and, hence, (i) follows. Using (3.15) we have that $q'(x) \geq 0 \forall x \geq 0$ if and only if $m'(y) \leq 0$ for $y \in (0, c/2)$. Furthermore, it is straightforward to check that

$$\alpha + \beta(\alpha) > 0 \Leftrightarrow a + b(a) := F^{-1}(\alpha) + F^{-1}(\alpha + d) > 0 \Leftrightarrow a > -\bar{a}.$$

From (3.16), for $n = 1$, and (3.4) we deduce that

$$k_1((\alpha, \alpha + d); m) = \lambda_1((a, b(a)); q^{-1}) = \mu_1((a, b(a)); q) =: \mu_1(a).$$

The above chain of equalities immediately implies that

$$\frac{d}{d\alpha} k_1((\alpha, \alpha + d); m) = \frac{1}{q(a)} \frac{d}{da} \mu_1(a),$$

which in turn, thanks to Lemma 3.4, yields (ii) and (iii). \square

Lemma 3.2 also allows to obtain qualitative properties about the first nontrivial eigenfunction to problem (3.1).

Lemma 3.5. *Let q be even and u_1 an eigenfunction to problem (3.1) with eigenvalue μ_1 and $a + b > 0$.*

(i) *If $q'(x) \geq 0$ for $x > 0$, then*

$$|u_1(a)| \geq |u_1(b)|. \quad (3.31)$$

(ii) *If $q'(x) \leq 0$ for $x > 0$, then*

$$|u_1(a)| \leq |u_1(b)|. \quad (3.32)$$

Moreover, inequalities (3.31) and (3.32) are strict if q is not constant on (a, b) .

Proof. By Lemma 3.1, we may assume that

$$u_1(x) = -\frac{v_1'(x)}{\lambda_1((a, b); q^{-1}) q(x)}, \quad (3.33)$$

where v_1 is an eigenfunction to problem (3.3) corresponding to λ_1 . Since $w_1(F(x)) = Cv_1(x)$ for some constant $C \neq 0$, where $w_1(y)$ is the function defined in Lemma 3.2, identity (3.33) becomes

$$u_1(F^{-1}(y)) = -\frac{1}{C} \frac{w_1'(y)}{\lambda_1((a, b); q^{-1})} = -\frac{1}{C} \frac{w_1'(y)}{k_1((\alpha, \beta); m)}.$$

Now the assertions follow from Lemma 3.2. \square

4. THE N -DIMENSIONAL CASE

Let us consider the problem (1.1) in B_R , the ball centered at the origin with radius R , i.e.

$$\begin{cases} -\operatorname{div}(e^h \nabla u) = \mu e^h u & \text{in } B_R \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_R. \end{cases} \quad (4.1)$$

The equation in (4.1) can be rewritten, using polar coordinates, as

$$\frac{1}{r^{N-1}} \frac{\partial}{\partial r} \left(r^{N-1} \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \Delta_{\mathbb{S}^{N-1}} (u|_{\mathbb{S}_r^{N-1}}) + h'(r) \frac{\partial u}{\partial r} + \mu u = 0, \quad (4.2)$$

where \mathbb{S}_r^{N-1} is the sphere of radius r in \mathbb{R}^N , $u|_{\mathbb{S}_r^{N-1}}$ is the restriction of u on \mathbb{S}_r^{N-1} and finally $\Delta_{\mathbb{S}^{N-1}} (u|_{\mathbb{S}_r^{N-1}})$ is the standard Laplace-Beltrami operator relative to the manifold \mathbb{S}_r^{N-1} .

Looking for separated solutions $u(x) = Y(\theta) f(r)$ of equation (4.2), where θ belongs to \mathbb{S}_1^{N-1} , we find

$$Y \frac{1}{r^{N-1}} (r^{N-1} f')' + \Delta_{\mathbb{S}^{N-1}} Y \frac{f}{r^2} + Y h'(r) f' + \mu Y f = 0,$$

and hence

$$\frac{1}{r^{N-3} f} (r^{N-1} f')' + r^2 h'(r) \frac{f'}{f} + \mu r^2 = -\frac{\Delta_{\mathbb{S}^{N-1}} Y}{Y} = \bar{k}. \quad (4.3)$$

As well known, see, e.g., [24] and [12], the last equality is fulfilled if and only if

$$\bar{k} = k(k + N - 2) \quad \text{with } k \in \mathbb{N} \cup \{0\}. \quad (4.4)$$

Multiplying the left hand side of equation (4.3) by $\frac{f}{r^2}$, we get

$$f'' + f' \left(\frac{N-1}{r} + h'(r) \right) + \mu f - k(k + N - 2) \frac{f}{r^2} = 0 \quad \text{in } (0, R).$$

Let us denote by f_k, Y_k the solutions of (4.3) with $k = \bar{k}$ defined in (4.4).

The eigenfunctions are either purely radial

$$u_i(r) = f_0(\mu_i; r), \quad \text{if } k = 0, \quad (4.5)$$

or have the form

$$u_i(r, \theta) = f_k(\mu_i; r) Y_k(\theta), \quad \text{if } k \in \mathbb{N}. \quad (4.6)$$

The functions f_k , with $k \in \mathbb{N} \cup \{0\}$, clearly satisfy

$$\begin{cases} f_k'' + f_k' \left(\frac{N-1}{r} + h'(r) \right) + \mu_i f_k - k(k+N-2) \frac{f_k}{r^2} = 0 & \text{in } (0, R) \\ f_k(0) = 0, \quad f_k'(R) = 0. \end{cases} \quad (4.7)$$

In the sequel we will denote by $\tau_n(R)$, with $n \in \mathbb{N} \cup \{0\}$, the increasing sequence of eigenvalues of (4.1) whose corresponding eigenfunctions are purely radial, i.e. in the form (4.5) or equivalently solutions to problem (4.7) with $k = 0$. Clearly in this case the first eigenfunction is constant and the corresponding eigenvalue $\tau_0(R)$ is trivially zero. We will denote by $\nu_n(R)$, with $n \in \mathbb{N}$, the remaining eigenvalues of (4.1), arranged in increasing order.

Lemma 4.1. *If the function $h(r)$ fulfills assumptions (1.5) then*

$$\nu_1(R) < \tau_1(R), \quad \forall R > 0. \quad (4.8)$$

Proof. We recall that $\tau_1 = \tau_1(R)$ is the first nontrivial eigenvalue of

$$\begin{cases} g'' + g' \left(\frac{N-1}{r} + h'(r) \right) + \tau g = 0 & \text{in } (0, R) \\ g'(0) = g'(R) = 0, \end{cases} \quad (4.9)$$

and $\nu_1 = \nu_1(R)$ is the first eigenvalue of

$$\begin{cases} w'' + \left(\frac{N-1}{r} + h'(r) \right) w' + \nu w - \frac{N-1}{r^2} w = 0 & \text{in } (0, R) \\ w(0) = w'(R) = 0. \end{cases} \quad (4.10)$$

First of all we observe that the first eigenfunction w_1 of (4.10) does not change its sign in $(0, R)$, thus we can assume that $w_1 > 0$ in $(0, R)$.

Moreover $w_1' > 0$ in $(0, R)$. Indeed, assume, by contradiction, that we can find two values r_1, r_2 , with $r_1 < r_2$, such that $w_1''(r_1) \leq 0$, $w_1'(r_1) = 0$ and $w_1''(r_2) \geq 0$, $w_1'(r_2) = 0$. By evaluating the equation in (4.10)

$$\frac{w_1''}{w_1} + \frac{w_1'}{w_1} \left(\frac{N-1}{r} + h'(r) \right) + \nu_1 - \frac{N-1}{r^2} = 0$$

at r_1 and r_2 , we get

$$\nu_1 - \frac{N-1}{r_2^2} \leq 0 \quad \text{and} \quad \nu_1 - \frac{N-1}{r_1^2} \geq 0,$$

which means $r_1 \geq r_2$ and this is a contradiction.

On the other hand, the first nontrivial eigenfunction of problem (4.9), $g_1 = g_1(r)$, has mean value zero i.e.

$$\int_{B_R} g_1 e^{h(|x|)} dx = N\omega_N \int_0^R g_1(r) e^{h(r)} r^{N-1} dr = 0,$$

where, here and in the sequel, ω_N denotes the Lebesgue measure of the unit ball in \mathbb{R}^N .

This implies that $g_1(r)$ must change its sign in $(0, R)$. Let us suppose $g_1(r) > 0$ in $(0, r_0)$ and $g_1(r_0) = 0$. We observe that $g_1'(r) < 0$ in $(0, R)$ and in particular

$$g_1'(r_0) < 0. \quad (4.11)$$

Therefore evaluating the equation of problem (4.9) at r_0 , we have

$$g_1''(r_0) + g_1'(r_0) \left(\frac{N-1}{r_0} + h'(r_0) \right) = 0 \quad (4.12)$$

and by the assumption on h' , see (1.5), it follows that

$$g_1''(r_0) > 0. \quad (4.13)$$

Moreover if we set $\psi = g_1'$, then problem (4.9) becomes

$$\begin{cases} \psi'' + \psi' \left(\frac{N-1}{r} + h'(r) \right) + \psi \left(-\frac{N-1}{r^2} + h''(r) \right) + \tau_1 \psi = 0 & \text{in } (0, R) \\ \psi(0) = \psi(R) = 0. \end{cases} \quad (4.14)$$

Further since we are assuming, see (1.5), that $h'' \geq 0$, from (4.10) we have that

$$\begin{cases} w'' + \left(\frac{N-1}{r} + h'(r) \right) w' + (\nu_1 + h''(r))w - \frac{N-1}{r^2}w \geq 0 & \text{in } (0, R) \\ w(0) = w(R) = 0. \end{cases} \quad (4.15)$$

Now we multiply the equation in (4.14) by $r^{N-1}e^{h(r)}w_1$ and the equation in (4.15) by $r^{N-1}e^{h(r)}\psi$, respectively, and, finally, subtracting, leads to

$$\begin{aligned} & r^{N-1}e^{h(r)}(w_1''\psi - w_1\psi'') + r^{N-1}e^{h(r)}\left(\frac{N-1}{r} + h'(r)\right)(w_1'\psi - w_1\psi') + \\ & + (\nu_1 - \tau_1)r^{N-1}e^{h(r)}w_1\psi \geq 0 \quad \text{in } (0, r_0). \end{aligned}$$

Integrating the above inequality on $(0, r_0)$, we get

$$\begin{aligned} & (\nu_1 - \tau_1) \int_0^{r_0} w_1\psi r^{N-1}e^{h(r)} dr \geq \\ & \int_0^{r_0} \left[(w_1\psi'' - w_1''\psi) + \left(\frac{N-1}{r} + h'(r) \right) (w_1\psi' - w_1'\psi) \right] e^{h(r)} r^{N-1} dr. \end{aligned} \quad (4.16)$$

Now we claim that

$$\int_0^{r_0} \left[(w_1 \psi'' - w_1'' \psi) + \left(\frac{N-1}{r} + h'(r) \right) (w_1 \psi' - w_1' \psi) \right] e^{h(r)} r^{N-1} dr > 0$$

To this aim we first note that

$$\begin{aligned} \int_0^{r_0} \psi'' w_1 r^{N-1} e^{h(r)} dr &= r_0^{N-1} \psi'(r_0) w_1(r_0) e^{h(r_0)} \\ &\quad - \int_0^{r_0} \psi' \left(h'(r) w_1 + w_1' + \frac{N-1}{r} w_1 \right) e^{h(r)} r^{N-1} dr \end{aligned} \quad (4.17)$$

and

$$\begin{aligned} \int_0^{r_0} w_1'' \psi r^{N-1} e^{h(r)} dr &= r_0^{N-1} \psi(r_0) w_1'(r_0) e^{h(r_0)} \\ &\quad - \int_0^{r_0} w_1' \left(h'(r) \psi + \psi' + \frac{N-1}{r} \psi \right) e^{h(r)} r^{N-1} dr. \end{aligned} \quad (4.18)$$

Recalling that $\psi'(r_0) = g_1''(r_0) > 0$, see (4.13), and $\psi(r_0) = g_1'(r_0) < 0$, see (4.11), we have

$$r_0^{N-1} e^{h(r_0)} (\psi'(r_0) w_1(r_0) - \psi(r_0) w_1'(r_0)) > 0. \quad (4.19)$$

Hence, subtracting equations (4.17) and (4.18), taking into account of (4.19), from (4.16) we get

$$(\nu_1 - \tau_1) \int_0^{r_0} w_1 \psi e^{h(r)} r^{N-1} dr > 0.$$

Finally, since $\psi(r) = g_1'(r) < 0$ in $(0, R)$ and $w_1(r) > 0$ in $(0, R)$, we have that

$$\int_0^{r_0} w_1 \psi e^{h(r)} r^{N-1} dr < 0,$$

and therefore $(\nu_1 - \tau_1)$ must be negative. The Lemma is so proved. \square

From Lemma 4.1 we clearly have

$$\mu_1(B_R; e^{h(|x|)}) = \frac{\int_{B_R} \left((w'(|x|))^2 + \frac{N-1}{|x|^2} w(|x|)^2 \right) d\gamma_h}{\int_{B_R} w(|x|)^2 d\gamma_h}, \quad \forall R > 0. \quad (4.20)$$

Now we are in position to prove the main result of this Section.

Proof of Theorem 1.2 Recall that Ω^\star is the ball B_{r^\star} such that $\gamma_h(\Omega^\star) = \gamma_h(B_{r^\star})$. We define

$$G(r) = \begin{cases} w(r) & \text{for } 0 < r < r^\star \\ w(r^\star) & \text{for } r \geq r^\star, \end{cases} \quad (4.21)$$

where w is the solution to problem (4.10) satisfying (4.20). By the results stated above the function G is nondecreasing and nonnegative. We introduce the functions

$$P_i(x) = G(|x|) \frac{x_i}{|x|} \quad \text{for } 1 \leq i \leq N.$$

The assumption on the symmetry of Ω guarantees

$$\int_{\Omega} P_i(x) d\gamma_h = 0, \quad \forall i = 1, \dots, N. \quad (4.22)$$

Hence each function P_i is admissible in the variational formulation of $\mu_1(\Omega; e^{h(|x|)})$, i.e. (2.4).

Since

$$\frac{\partial P_i}{\partial x_j} = G'(|x|) \frac{x_i x_j}{|x|^2} - G(|x|) \frac{x_i x_j}{|x|^3} + \delta_{ij} \frac{G(|x|)}{|x|},$$

where δ_{ij} is the Kronecker symbol. Using P_i as trial functions for $\mu_1(\Omega; e^{h(|x|)})$ we get

$$\mu_1(\Omega; e^{h(|x|)}) \leq \frac{\sum_{i=1}^N \int_{\Omega} \sum_{j=1}^N \left(\frac{\partial P_i}{\partial x_j} \right)^2 d\gamma_h}{\sum_{i=1}^N \int_{\Omega} P_i^2 d\gamma_h} = \frac{\int_{\Omega} N(|x|) d\gamma_h}{\int_{\Omega} D(|x|) d\gamma_h}. \quad (4.23)$$

where

$$N(r) = (G'(r))^2 + \frac{N-1}{r^2} G^2(r)$$

and

$$D(r) = G^2(r).$$

We claim that

$$\frac{d}{dr} N(r) < 0.$$

Taking into account the definition (4.21) of G , and the differential equation in (4.10), with $\nu = \nu_1$, we have

$$\frac{d}{dr} N(r) = \begin{cases} -2 \left[\nu_1 w w' + (w')^2 r + \frac{N-1}{r^3} (r w' + w)^2 \right] & \text{if } 0 < r < r^{\star} \\ -2(N-1) \frac{w^2(r^{\star})}{(r^{\star})^3} & \text{if } r \geq r^{\star} \end{cases}$$

Now we claim that

$$\int_{\Omega} N(|x|) d\gamma_h \leq \int_{\Omega^{\star}} N(|x|) d\gamma_h. \quad (4.24)$$

Hardy-Littlewood inequality (2.1) ensures

$$\int_{\Omega} N(|x|) d\gamma_h \leq \int_0^{\gamma_h(\Omega)} N^*(s) ds, \quad (4.25)$$

where N^* is the decreasing rearrangement of N . Setting

$$s = \gamma_h(B_r) = N\omega_N \int_0^r e^{h(s)} s^{N-1} ds,$$

we get

$$\int_0^{\gamma_h(\Omega)} N^*(s) ds = N\omega_N \int_0^{r^{\star}} N^*(\gamma_h(B_r)) e^{h(r)} r^{N-1} dr.$$

Note that

$$N^*(\gamma_h(B_r)) = N(r),$$

since $N^*(\gamma_h(B_r))$ and $N(r)$ are equimeasurable and both radially decreasing functions. Therefore

$$N\omega_N \int_0^{r^{\star}} N^*(\gamma_h(B_r)) e^{h(r)} r^{N-1} dr = N\omega_N \int_0^{r^{\star}} N(r) e^{h(r)} r^{N-1} dr = \int_{\Omega^{\star}} N(|x|) d\gamma_h \quad (4.26)$$

Combining (4.25) and (4.26), we obtain the claim (4.24). Analogously it is possible to prove that

$$\int_{\Omega} D(|x|) d\gamma_h \geq \int_{\Omega^{\star}} D(|x|) d\gamma_h. \quad (4.27)$$

Indeed since D is an increasing function, we have

$$\begin{aligned} \int_{\Omega} D(|x|) e^{h(|x|)} dx &\geq \int_0^{\gamma_h(\Omega)} D_*(s) ds \\ &= N\omega_N \int_0^{r^{\star}} D_*(\gamma_h(B_r)) e^{h(r)} r^{N-1} dr = \int_{\Omega^{\star}} D(|x|) d\gamma_h, \end{aligned}$$

where D_* is the increasing rearrangement of D . By (4.21), (4.24) and (4.27), inequality (4.23) implies

$$\mu_1(\Omega; e^{h(|x|)}) \leq \frac{\int_{\Omega^{\star}} \left((w'(|x|))^2 + \frac{N-1}{|x|^2} w(|x|)^2 \right) d\gamma_h}{\int_{\Omega^{\star}} w(|x|)^2 d\gamma_h} = \mu_1(\Omega^{\star}; e^{h(|x|)}),$$

which is our claim. Moreover, from the monotonicity properties of the functions N and D , it is easy to realize that inequalities (4.24) and (4.27) reduce to equalities only when Ω is the ball Ω^{\star} .

We finally exhibit an example showing that, in general, the condition about the symmetry of the domain cannot be dropped.

Let

$$H_n(t) := (-1)^n e^{t^2} \left(\frac{d^n}{dt^n} e^{-t^2} \right), \quad t \in \mathbb{R},$$

and

$$v_n(t) := H_n(t) e^{-t^2}, \quad t \in \mathbb{R}.$$

Let c and d be the first and second positive zeros of $v'_5(t) = -8e^{-t^2} (8t^6 - 60t^4 + 90t^2 - 15)$ respectively. It is elementary to verify that

$$c \in (0.43, 0.44) \text{ and } d \in (1.33, 1.34). \quad (4.28)$$

We consider the following two-dimensional problem with anti-Gaussian degeneracy

$$\begin{cases} -\operatorname{div} \left(e^{x^2+y^2} \nabla u \right) = \mu e^{x^2+y^2} u & \text{in } T \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial T, \end{cases}$$

where $T = (c, d) \times (-c, c)$. A straightforward computation shows that $\mu_1(T) = 12$, moreover $\mu_1(T)$ is a double eigenvalue and a corresponding set of independent eigenfunctions are $u_1(x, y) := v_5(x)$ and $u_2(x, y) := v_5(y)$ (see, e.g., [28], p.104 ff.).

Define

$$d\gamma_2 = e^{x^2+y^2} dx dy, \quad \text{with } (x, y) \in \mathbb{R}^2.$$

Now we claim that the ball B_{r_T} , such that $\gamma_2(B_{r_T}) = \gamma_2(T)$, fulfills

$$\mu_1(B_{r_T}; \gamma_2) < 12 = \mu_1(T; \gamma_2). \quad (4.29)$$

Clearly

$$\gamma_2(B_r) = \pi \left(e^{r^2} - 1 \right) =: \chi(r), \quad \text{with } r > 0,$$

and therefore

$$r_T = \chi^{-1}(\gamma_2(T)).$$

As recalled in Section 2, $\mu_1(B_{r_T}, \gamma_2)$ satisfies the following variational characterization

$$\mu_1(B_{r_T}; \gamma_2) = \min \left\{ \frac{\int_{B_{r_T}} |Dv|^2 d\gamma_2}{\int_{B_{r_T}} v^2 d\gamma_2} : v \in H^1(B_{r_T}) \setminus \{0\}, \int_{B_{r_T}} v d\gamma_2 = 0 \right\} \quad (4.30)$$

In order to get an estimate from above for $\mu_1(B_{r_T}; \gamma_2)$, we use $v = x$ and $v = y$ as trial functions in (4.30) obtaining

$$\mu_1(B_{r_T}; \gamma_2) < \frac{\gamma_2(B_{r_T})}{\int_{B_{r_T}} x^2 d\gamma_2} \quad \text{and} \quad \mu_1(B_{r_T}; \gamma_2) < \frac{\gamma_2(B_{r_T})}{\int_{B_{r_T}} y^2 d\gamma_2}.$$

Summing up we get

$$\mu_1(B_{r_T}; \gamma_2) < \frac{2 \int_0^{r_T} e^{s^2} s ds}{\int_0^{r_T} e^{s^2} s^3 ds} = k(r_T), \quad (4.31)$$

where

$$k(r) := \frac{2e^{r^2} - 2}{r^2 e^{r^2} - e^{r^2} + 1}, \quad \text{with } r > 0.$$

A Taylor expansion of e^{x^2} allows to estimate from below $\gamma_2(T)$ as follows

$$\gamma_2(T) > \int_c^d \left(1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6}\right) dx \int_{-c}^c \left(1 + y^2 + \frac{y^4}{2} + \frac{y^6}{6}\right) dy > 2, \quad (4.32)$$

where the last inequality is an immediate consequence of (4.28).

Since χ^{-1} is an increasing function, by (4.32) we have

$$\chi^{-1}(\gamma_2(T)) > \chi^{-1}(2) = \sqrt{\log\left(1 + \frac{2}{\pi}\right)}.$$

Finally since k is a decreasing function, the above inequality together with (4.31) imply

$$\mu_1(B_{r_T}; \gamma_2) < k(r_T) < k(\chi^{-1}(2)) = \frac{4}{(\pi + 2) \log(1 + \frac{2}{\pi}) - 2} < 12.$$

Hence the claim is proved. □

Remark 4.1. *Note that the assumption on the symmetry of Ω is used solely to guarantee the orthogonality conditions (4.22).*

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